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Four positive solutions for a semilinear elliptic equation

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0. Introduction

This paper is based on the joint work [**AT1**] with K. Tanaka. In this paper, we study the existence and multiplicity of positive solutions of the following semilinear elliptic equation:

$$\begin{cases} -\Delta u + u = a(x)u^p + f(x) & \text{in } \mathbf{R}^N, \\ \quad u > 0 & \text{in } \mathbf{R}^N, \\ \quad u \in H^1(\mathbf{R}^N), \end{cases} \tag{0.1}$$

where $1 < p < \dfrac{N+2}{N-2}$ $(N \geq 3)$, $1 < p < \infty$ $(N = 1, 2)$, $a(x) \in C(\mathbf{R}^N)$, $f(x) \in H^{-1}(\mathbf{R}^N)$ and $f(x) \geq 0$. We also assume that

(H1) $a(x) > 0$ for all $x \in \mathbf{R}^N$,
(H2) $a(x) \to 1$ as $|x| \to \infty$,
(H3) there exist $\delta > 0$ and $C > 0$ such that

$$a(x) - 1 \geq -Ce^{-(2+\delta)|x|} \qquad \text{for all } x \in \mathbf{R}^N.$$

(H4) $a(x) \in (0, 1]$ for all $x \in \mathbf{R}^N$, $a(x) \not\equiv 1$.

First of all, we consider in the case $f(x) \equiv 0$:

$$\begin{cases} -\Delta u + u = a(x)u^p & \text{in } \mathbf{R}^N, \\ \quad u > 0 & \text{in } \mathbf{R}^N, \\ \quad u \in H^1(\mathbf{R}^N). \end{cases} \tag{0.2}$$

Positive solutions of (0.2) are corresponding to certain kinds of standing waves in nonlinear equations of the Schrödinger or Klein-Gordon type. The existence of positive solutions of (0.2) depends on the shape of $a(x)$ delicately. For example, in the case $a(x) \equiv 1$:

$$\begin{cases} -\Delta u + u = u^p & \text{in } \mathbf{R}^N, \\ \quad u > 0 & \text{in } \mathbf{R}^N, \\ \quad u \in H^1(\mathbf{R}^N), \end{cases} \tag{0.3}$$

it is known that the equation (0.3) has a unique positive radial solution $\omega(x) = \omega(|x|) > 0$ and any positive solution $u(x)$ of (0.3) can be written as

$$u(x) = \omega(x - x_0) \quad \text{for some } x_0 \in \mathbf{R}^N .$$

(See Kwong [**K**], c.f. Kabeya-Tanaka [**KT**]).

In the case $a(x) \not\equiv 1$, the situation is completely different even if the difference between $a(x)$ and 1 is small. (c.f. Lions [**PLL1**, **PLL2**]). For example, if $a(x)$ satisfies

$$a(x) \geq 1 \quad \text{for all } x \in \mathbf{R}^N, \tag{0.4}$$

then we can see that the minimax value given by the Mountain Pass Theorem — we call it the MP level in short — is lower than the first level of breaking down of the Palais-Smale condition. Thus we can obtain a positive solution of (0.2) via the Mountain Pass Theorem. On the other hand, if $a(x)$ satisfies (H4), then we can see that the MP level is exactly equal to the first level of breaking down of the Palais-Smale condition and we can not get a positive solution through the Mountain Pass Theorem.

We remark that Bahri-Li [**BaYL**] showed that the existence of at least one positive solution of (0.2) only under (H1)–(H3). See also Bahri-Lions [**BaPLL**], in which they showed the existence of at least one positive solution under condition $N \geq 2$ and

$$a(x) - 1 \geq -C \exp(-\delta|x|) \quad \text{for all } x \in \mathbf{R}^N .$$

Here we study for the case $f(x) \geq 0$, $f(x) \not\equiv 0$. Our main question is whether positive solutions can survive after a perturbation of type (0.1) or not. Such a question was studied by Zhu [**Z**], Cao-Zhou [**CZ**], Jeanjean [**J**], Hirano [**H**] and Adachi-Tanaka [**AT2**]. See also Ambrosetti and Badiale [**AB**] for a perturbation result via Poincaré-Melnikov type arguments. Zhu [**Z**] (c.f. Hirano [**H**]) were mainly concerned with the case $a(x) \equiv 1$ and $f(x) \geq 0$, $f(x) \not\equiv 0$ and succeeded to find the existence of at least two positive solutions under the situation

$$\|f\|_{H^{-1}(\mathbf{R}^N)} \leq M, \tag{0.5}$$

where the constant $M > 0$ was chosen so that the corresponding functional:

$$I(u) = \frac{1}{2}\int_{\mathbf{R}^N} |\nabla u|^2 + |u|^2\, dx - \frac{1}{p+1}\int_{\mathbf{R}^N} u^{p+1}\, dx - \int_{\mathbf{R}^N} fu\, dx$$

possesses the mountain pass environment. That is, there exist $\delta_0 > 0$, $\rho_0 > 0$ and $e \in H^1(\mathbf{R}^N)$ such that

$$I(u) \geq \delta_0 \qquad \text{for all } \|u\|_{H^1(\mathbf{R}^N)} = \rho_0$$

and

$$\|e\|_{H^1(\mathbf{R}^N)} > \rho_0, \ I(e) < 0.$$

Generalizations of the result of [**Z**] were done by Cao-Zhou [**CZ**], Jeanjean [**J**] and Adachi-Tanaka [**AT2**]. They studied more general nonlinearities

$$\begin{cases} -\Delta u + u = g(x,u) + f(x) & \text{in } \mathbf{R}^N, \\ \quad u > 0 & \text{in } \mathbf{R}^N, \\ \quad u \in H^1(\mathbf{R}^N), \end{cases} \tag{0.6}$$

under suitable conditions. [**CZ**] and [**J**] showed the existence of at least two positive solutions especially under the assumption:

$$g(x,u) \geq \overline{g}(u) \left(= \lim_{|x|\to\infty} g(x,u) \right) \qquad \text{for all } x \in \mathbf{R}^N \text{ and } u > 0. \tag{0.7}$$

The assumption (0.7) is corresponding to (0.4). When $f \equiv 0$, by the concentration compactness principle, we also see that the Mountain Pass Theorem works under the assumption (0.7). Thus the assumption (0.7) makes it easy to study (0.6) via variational methods.

In this paper, we study the multiplicity of positive solutions of (0.1) under the assumption (H4). The situation is completely different from [**CZ**], [**J**] and as far as we know, such a situation has not been studied. Technical difficulty is also different. For instance, we use Lusternik-Schnirelman category instead of Mountain Pass Theorem to show the existence of positive solutions of (0.1) and we show the existence of more positive solutions under the assumption (H4). Our main results are the following

Theorem 0.1 ([AT1]). *Assume (H1)–(H4). Then there exists a $\delta_0 > 0$ such that for non-negative function $f(x)$ satisfying $0 < \|f\|_{H^{-1}(\mathbf{R}^N)} \leq \delta_0$, (0.1) possesses at least four positive solutions.*

As to an asymptotic behavior of solutions obtained in Theorem 0.1 as $\|f\|_{H^{-1}(\mathbf{R}^N)} \to 0$, we have

Theorem 0.2 ([AT1]). *Assume that a sequence of non-negative functions $(f_j(x))_{j=1}^\infty \subset H^{-1}(\mathbf{R}^N)$ satisfies $f_j(x) \not\equiv 0$ and*

$$\|f_j\|_{H^{-1}(\mathbf{R}^N)} \to 0 \qquad \text{as } j \to \infty.$$

Then there exist a subsequence of $(f_j(x))_{j=1}^\infty$ — still denoted by $(f_j(x))_{j=1}^\infty$ — and four sequences $(u_j^{(k)}(x))_{j\in\mathbf{N}}$ $(k = 1,2,3,4)$ of positive solutions of (0.1) with $f(x) = f_j(x)$ such

that
(i) $\|u_j^{(1)}\|_{H^1(\mathbf{R}^N)} \to 0$ *as* $j \to \infty$.
(ii) *There exist sequences* $(y_j^{(2)})_{j=1}^{\infty}$, $(y_j^{(3)})_{j=1}^{\infty} \subset \mathbf{R}^N$ *such that*

$$|y_j^{(k)}| \to \infty, \quad \|u_j^{(k)}(x) - \omega(x - y_j^{(k)})\|_{H^1(\mathbf{R}^N)} \to 0$$

as $j \to \infty$ *for* $k = 2, 3$.
(iii) *There exists a positive solution* $v_0(x)$ *of* (0.2) *such that*

$$\|u_j^{(4)}(x) - v_0(x)\|_{H^1(\mathbf{R}^N)} \to 0 \qquad \text{as } j \to \infty.$$

We remark that the solutions $u^{(2)}(x)$, $u^{(3)}(x)$ do not converge strongly to solutions of (0.1) with $f \equiv 0$. As an immediate corollary to Theorem 0.2, we have the following result on symmetry-breaking of positive solutions for (0.1).

Corollary 0.3 ([AT1]). *Suppose that* $a(x) = a(|x|)$, $f(x) = f(|x|)$ *are radially symmetric in addition to (H1)–(H4). Then there exists a* $\delta_1 > 0$ *such that if* $f(x) \geq 0$, $f(x) \not\equiv 0$, $\|f\|_{H^{-1}(\mathbf{R}^N)} \leq \delta_1$, *then* (0.1) *possesses at least one positive solution which is not radially symmetric.*

In next Section, we sketch the proof of Theorem 0.1.

1. Outline of the proof of Theorem 0.1

We use variational methods to find positive solutions of (0.1). We divide outline of the proof of Theorem 0.1 into several steps.

Step 1 : functional setting

We define for given $a(x)$ and $f(x)$

$$I_{a,f}(u) = \frac{1}{2}\|u\|_{H^1(\mathbf{R}^N)}^2 - \frac{1}{p+1}\int_{\mathbf{R}^N} a(x)u_+^{p+1}\,dx - \int_{\mathbf{R}^N} fu\,dx : H^1(\mathbf{R}^N) \to \mathbf{R},$$
$$J_{a,f}(v) = \max_{t>0} I_{a,f}(tv) : \Sigma_+ \to \mathbf{R},$$

where

$$\|u\|_{H^1(\mathbf{R}^N)} = \left(\int_{\mathbf{R}^N} (|\nabla u|^2 + |u|^2)\,dx\right)^{\frac{1}{2}},$$
$$\Sigma = \{v \in H^1(\mathbf{R}^N)\,;\, \|v\|_{H^1(\mathbf{R}^N)} = 1\},$$
$$\Sigma_+ = \{v \in \Sigma\,;\, v_+ \not\equiv 0\}.$$

We will see that critical points of $I_{a,f}(u) : H^1(\mathbf{R}^N) \to \mathbf{R}$ or $J_{a,f}(v) : \Sigma_+ \to \mathbf{R}$ are corresponding to positive solutions of (0.1). We remark that if $\|f\|_{H^{-1}(\mathbf{R}^N)}$ is sufficiently small, then $I_{a,f}(u)$ has a mountain pass geometry, that is, $I_{a,f}(u)$ satisfies

(i) there exists a constant $\rho_0 > 0$ such that

$$I_{a,f}(u) \geq 0 \quad \text{for all } u \in H^1(\mathbf{R}^N) \text{ with } \|u\|_{H^1(\mathbf{R}^N)} = \rho_0,$$

(ii) $\{u \in H^1(\mathbf{R}^N)\,;\, \|u\|_{H^1(\mathbf{R}^N)} > \rho_0 \text{ and } I_{a,f}(u) < 0\} \neq \emptyset$,

(iii) $\inf_{\|u\|_{H^1(\mathbf{R}^N)} < \rho_0} I_{a,f}(u) < 0.$

Step 2 : critical point near 0

First we find one positive solution $u^{(1)}(a, f; x) = u_{loc\ min}(a, f; x)$ as a local minimum of $I_{a,f}(u)$ in B_{ρ_0}, where $B_{\rho_0} = \{u \in H^1(\mathbf{R}^N)\,;\, \|u\|_{H^1(\mathbf{R}^N)} < \rho_0\}$. We see that there exists a critical point $u_{loc\ min}(a, f; x)$ satisfying

$$I_{a,f}(u_{loc\ min}) = \inf_{\|u\|_{H^1(\mathbf{R}^N)} < \rho_0} I_{a,f}(u) < 0.$$

We also see that $I_{a,f}(u_{loc\ min})$ is the lowest functional level among all positive solutions of (0.1). Moreover it is easily seen that

$$u_{loc\ min}(a, f; x) \to 0 \quad \text{in } H^1(\mathbf{R}^N) \text{ as } \|f\|_{H^{-1}(\mathbf{R}^N)} \to 0.$$

Thus $u_{loc\ min}(a, f; x)$ is the solution of (0.1) which satisfies the property (i) in Theorem 0.2.

Step 3 : breaking down of the Palais-Smale condition

We study here the breaking down of the Palais-Smale condition for $I_{a,f}(u)$. The unique positive radial solution $\omega(x)$ of the limit equation (0.3) plays an important role to describe an asymptotic behavior of the Palais-Smale sequence for $I_{a,f}(u)$.

Definition. For $c \in \mathbf{R}$ we say that $(u_j)_{j=1}^{\infty} \subset H^1(\mathbf{R}^N)$ is a *(PS)$_c$-sequence* for $I_{a,f}(u)$, if and only if $(u_j)_{j=1}^{\infty}$ satisfies

$$\begin{aligned} &I_{a,f}(u_j) \to c, \\ &I'_{a,f}(u_j) \to 0 \quad \text{in } H^{-1}(\mathbf{R}^N), \end{aligned}$$

as $j \to \infty$. We also say $I_{a,f}(u)$ satisfies *(PS)$_c$-condition* if any (PS)$_c$-sequence possesses a strongly convergent subsequence in $H^1(\mathbf{R}^N)$.

Proposition 1.1. *Assume that (H1)–(H4) and suppose that $(u_j)_{j=1}^{\infty} \subset H^1(\mathbf{R}^N)$ is a (PS)$_c$-sequence for $I_{a,f}(u)$. Then there exist a subsequence — still we denote by $(u_j)_{j=1}^{\infty}$*

—, *a critical point* $u_0(x)$ *of* $I_{a,f}(u)$, *an integer* $\ell \in \mathbf{N} \cup \{0\}$, *and* ℓ *sequences of points* $(y_j^1)_{j=1}^\infty, \ldots, (y_j^\ell)_{j=1}^\infty \subset \mathbf{R}^N$ *such that*

1° $|y_j^k| \to \infty$ *as* $j \to \infty$ *for all* $k = 1, 2, \ldots, \ell$,

2° $|y_j^k - y_j^{k'}| \to \infty$ *as* $j \to \infty$ *for* $k \neq k'$,

3° $\left\| u_j(x) - \left(u_0(x) + \sum_{k=1}^{\ell} \omega(x - y_j^k) \right) \right\|_{H^1(\mathbf{R}^N)} \to 0$ *as* $j \to \infty$,

4° $I_{a,f}(u_j) \to I_{a,f}(u_0) + \ell I_{1,0}(\omega)$ *as* $j \to \infty$.

This is rather standard result. See [**PLL1, PLL2**] for analogous arguments. From Proposition 1.1, we see that $(\mathrm{PS})_c$-condition breaks down only for

$$c = I_{a,f}(u_0) + \ell I_{1,0}(\omega),$$

where $u_0 \in H^1(\mathbf{R}^N)$ is a critical point of $I_{a,f}(u)$ and $\ell \in \mathbf{N}$. In particular, $(\mathrm{PS})_c$-condition holds for the level

$$c \in (-\infty, I_{a,f}(u_{loc\ min}) + I_{1,0}(\omega)). \tag{1.1}$$

We remark that $(\mathrm{PS})_c$-sequence of $J_{a,f}(v)$ also satisfies similar asymptotic behavior as that of $I_{a,f}(u)$ and $(\mathrm{PS})_c$-condition of $J_{a,f}(v)$ also holds for the level (1.1).

Step 4 : Lusternik-Schnirelman category

In this Step, we find two positive solutions different from $u_{loc\ min}$ under the level $I_{a,f}(u_{loc\ min}) + I_{1,0}(\omega)$. We use notation:

$$[J_{a,f} \le c] = \{u \in \Sigma_+ \, ; \, J_{a,f}(u) \le c\}$$

for $c \in \mathbf{R}$. We will observe that for sufficiently small $\varepsilon > 0$

$$[J_{a,f} \le I_{a,f}(u_{loc\ min}(a, f; x)) + I_{1,0}(\omega) - \varepsilon]$$

is not empty and

$$\mathrm{cat}([J_{a,f} \le I_{a,f}(u_{loc\ min}(a, f; x)) + I_{1,0}(\omega) - \varepsilon]) \ge 2 \tag{1.2}$$

provided $f(x) \ge 0$, $f(x) \not\equiv 0$ and $\|f\|_{H^{-1}(\mathbf{R}^N)}$ is sufficiently small. Here $\mathrm{cat}(\cdot)$ stands for the Lusternik-Schnirelman category. As a consequence of (1.1) and (1.2), we find two positive solutions $u^{(2)}(a, f; x)$ and $u^{(3)}(a, f; x)$ satisfying

$$I_{a,f}(u^{(k)}(a, f; x)) < I_{a,f}(u_{loc\ min}(a, f : x)) + I_{1,0}(\omega) \quad \text{for } k = 2, 3. \tag{1.3}$$

We remark that for $f \equiv 0$, we see that

$$u_{loc\ min}(a, 0; x) \equiv 0$$

and

$$[J_{a,0} \leq I_{a,0}(u_{loc\ min}(a, 0; x) + I_{1,0}(\omega)] = \emptyset \tag{1.4}$$

and (1.2) is the key of our proof. To get (1.2), we use the following interaction phenomenon.

Proposition 1.2. *Assume that (H1)–(H4) and suppose that $f \geq 0$, $f \not\equiv 0$. Then there exists $R_0 > 0$ such that*

$$I_{a,f}(u_{loc\ min} + t\omega(x - y)) < I_{a,f}(u_{loc\ min}) + I_{1,0}(\omega) \tag{1.5}$$

for all $|y| \geq R_0$ and $t \geq 0$.

This idea is originally used by Bahri-Li [**BaYL**]. See also Bahri-Lions [**BaPLL**], Bahri-Coron [**BaC**], Taubes [**T**]. We remark that (1.5) does not hold for $f \equiv 0$. In fact, if $f \equiv 0$, then $u_{loc\ min}(a, 0; x) = 0$ and

$$I_{a,0}(u_{loc\ min}(a, 0; x) + \omega(x - y)) = I_{a,0}(\omega(x - y)) > I_{1,0}(\omega).$$

Step 5 : a positive solution related to Bahri-Li's solution

To find the fourth positive solution, we adapt the minimax method of Bahri-Li [**BaYL**] to our functional $J_{a,f}(v)$. More precisely, we define

$$b_{a,f} = \inf_{\gamma \in \Gamma} \sup_{y \in \mathbf{R}^N} J_{a,f}(\gamma(y)),$$

where

$$\Gamma = \{\gamma \in C(\mathbf{R}^N, \Sigma_+)\,;\, \gamma(y) = \frac{\omega(\cdot - y)}{\|\omega\|_{H^1(\mathbf{R}^N)}} \quad \text{for large } |y|\}.$$

Then by Proposition 1.1, we will find a positive solution $u^{(4)}(a, f; x)$ corresponding to the minimax value $b_{a,f}$ which satisfies

$$b_{a,f} = I_{a,f}(u^{(4)}(a, f; x)) \geq I_{a,f}(u_{loc\ min}(a, f; x)) + I_{1,0}(\omega) \tag{1.6}$$

for sufficiently small $\|f\|_{H^{-1}(\mathbf{R}^N)}$. To show Theorem 0.2, we also use (1.3) and (1.6) in an essential way. ∎

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